

On decaying rate of a quantum state

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February 1, 2008

Abstract

Decaying rate of a quantum system investigated using the Fubini-Study definition of distance between states.

1 Introduction

In Dirac notation a pure state $|\psi(t)\rangle$ can be written in an N complex dimensional vector space

$$|\psi\rangle = \sum_{i=1}^N z_i |i\rangle, \quad (1)$$

where $|i\rangle$ belongs to a given orthonormal basis and z_i 's are complex numbers. In quantum mechanics, states differ only by an overall complex factor α are equivalent

$$\vec{z} = (z_1, z_2, \dots, z_n) \sim \alpha(z_1, z_2, \dots, z_n), \quad \alpha \in \mathcal{C}. \quad (2)$$

This recent relation is an equivalence relation on \mathcal{C}^N and the set of equivalence classes $[\vec{z}]$ is by definition the projective space \mathcal{CP}^N . The complex numbers in (2) are known as homogeneous coordinates.

there is a natural notion of distance on \mathcal{CP}^N called the Fubini-Study[1] distance or metric. The distance x between two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$

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according to Fubini-Study is given by

$$\cos^2(x) = \frac{|\langle \psi_1 | \psi_2 \rangle|^2}{\langle \psi_1 | \psi_1 \rangle \langle \psi_2 | \psi_2 \rangle} = \frac{|\vec{z}_1 \cdot \vec{z}_2|^2}{\vec{z}_1 \cdot \vec{z}_1 \vec{z}_2 \cdot \vec{z}_2}. \quad (3)$$

where

$$\begin{aligned} |\psi_1\rangle &= \sum_{i=1}^N z_{1i} |i\rangle, \\ |\psi_2\rangle &= \sum_{i=1}^N z_{2i} |i\rangle, \end{aligned} \quad (4)$$

and $\vec{z}_1 \equiv (z_{11}, z_{12}, \dots, z_{1N})$, $\vec{z}_2 \equiv (z_{21}, z_{22}, \dots, z_{2N}) \in \mathcal{C}^N$.

2 Decaying rate of a quantum state

Let $|\psi(0)\rangle$ be the initial state of an arbitrary quantum system described by a Hamiltonian \hat{H} , using (5), we want to find the distance between the initial state $|\psi(0)\rangle$ and the final state $|\psi(t)\rangle$ of the system during the time evolution under Hamiltonian \hat{H} . For this purpose we find the infinitesimal form of the equation (1). Let the normalized state in time t be $|\psi(t)\rangle$ then for an infinitesimal evolution

$$\begin{aligned} |\psi(t+dt)\rangle &= (1 - \frac{i}{\hbar} dt \hat{H}) |\psi(t)\rangle, \\ &= |\psi(t)\rangle - \frac{i}{\hbar} dt \hat{H} |\psi(t)\rangle, \end{aligned} \quad (5)$$

so

$$\langle \psi(t) | \psi(t+dt) \rangle = 1 - \frac{i}{\hbar} dt \langle \psi(t) | \hat{H} | \psi(t) \rangle. \quad (6)$$

From (5) and the assumption $\langle \psi(t) | \psi(t) \rangle = 1$, we have

$$\begin{aligned} \langle \psi(t+dt) | \psi(t+dt) \rangle &= (\langle \psi(t) | + \frac{i}{\hbar} dt \langle \psi(t) | \hat{H}) (\langle \psi(t) | - \frac{i}{\hbar} dt \hat{H} | \psi(t) \rangle), \\ &= 1 + \frac{(dt)^2}{\hbar^2} \langle \psi(t) | \hat{H}^2 | \psi(t) \rangle, \end{aligned} \quad (7)$$

now from (6) and (7) we find the infinitesimal form of (1)

$$dx = \frac{dt}{\hbar}(\Delta \hat{H})_\psi, \quad (8)$$

where $(\Delta \hat{H})_\psi = \sqrt{\langle \psi(t) | \hat{H}^2 | \psi(t) \rangle - (\langle \psi(t) | \hat{H} | \psi(t) \rangle)^2}$, is the uncertainty of energy in state $|\psi(t)\rangle$ and is time-independent for a time-independent Hamiltonian. According to (8), the velocity of decaying v_d of the state $|\psi(t)\rangle$, can be defined as

$$v_d = \frac{(\Delta \hat{H})_\psi}{\hbar}. \quad (9)$$

The decaying rate of a state $|\psi(t)\rangle$, can be defined via survival amplitude at time t given by [2]

$$A_t = \langle \psi(0) | \psi(t) \rangle, \quad (10)$$

the survival probability is $|A_t|^2$.

Using (1) and it's infinitesimal form (8), it is straightforward to show

$$|A_t| = \cos\left(\int_0^t \frac{(\Delta \hat{H})_\psi}{\hbar} dt\right), \quad (11)$$

when the system is closed, (time-independent \hat{H}), (11) becomes

$$|A_t| = \cos\left(\frac{t(\Delta \hat{H})_\psi}{\hbar}\right), \quad (12)$$

which can be compared with the Mandelstam-Tamm inequality [2]

$$|A_t| \geq \cos\left(\frac{t(\Delta \hat{H})_\psi}{\hbar}\right). \quad (13)$$

So during the evolution of a closed system, the survival probability of the state $|\psi(t)\rangle$, takes it's minimum value in any instant of time.

Decaying rate w of a state $|\psi(t)\rangle$, can be defined as

$$w = \frac{d}{dt}(1 - |A_t|^2), \quad (14)$$

so for $|A_t|$ given by (11), we have

$$w = \sin\left(\int_0^t \frac{2(\Delta \hat{H})_\psi}{\hbar} dt\right) \frac{(\Delta \hat{H})_\psi}{\hbar}, \quad (15)$$

and for a closed system

$$w = \sin\left(\frac{2t(\Delta \hat{H})_\psi}{\hbar}\right) \frac{(\Delta \hat{H})_\psi}{\hbar}. \quad (16)$$

References

- [1] T. W. B. Kibble, Comm. Math. Phys. 65, 189 (1979).
- [2] S. Luo and Z. Zhang, Lett. Math. Phys. 71, 1-11 (2005)